Critical indices from perturbation analysis of the Callan-Symanzik equation*

George A. Baker, Jr.†

Service de Physique Théorique, Centre d'Etudes Nucléaires-Saclay, BP No. 2, 91190 Gif-sur-Yvette, France and Theoretical Division, University of California, Los Alamos Scientific Laboratories, Los Alamos, New Mexico 87545

Bernie G. Nickel

Department of Physics, University of Guelph, Guelph, Ontario, Canada N1G 2W1

Daniel I. Meiron

Mathematical Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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Recent results giving both the asymptotic behavior and the explicit values of the leading-order perturbation-expansion terms in fixed dimension for the coefficients of the Callan-Symanzik equation are analyzed by the the Borel-Leroy, Padé-approximant method for the n-component ϕ^4 model. Estimates of the critical exponents for these models are obtained for $n = 0$, 1, 2, and 3 in three dimensions with a typical accuracy of a few one thousandths. In two dimensions less accurate results are obtained.

I. INTRODUCTION AND SUMMARY

The problems of the computation of the behavior of a physical system near its critical point has attracted considerable attention in recent years. One promising method for the calculations of these properties has been the renormalization-group approach introduced by Wilson[1] and substantially elaborated by others.[2-6] Until recently, it has not been made truly quantitative. In a previous brief publication,[7] we have shown how to implement a suggestion of Parisi[8] to base quantitative calculations on the perturbative calculation of the coefficients of the Callan[9]-Symanzik[10] equations in fixed dimensions. In that publication we introduced the Padé-Borel summation procedure in the context of this problem and reported the calculation of higher-order terms in the perturbation expansion of the coefficients. Specifically we considered the continuous-spin, three-dimensional Ising model with an $\exp(-As^4+Bs^2)$ spin-weight factor.

In this paper we have extended our calculations. We have added one more term to the series expansions and computed the terms for a more general model with n-component spins and spin-weight factor,

$$\exp\left[-A\left(\sum_{j=1}^{n} s_j^2\right)^2 + B\left(\sum_{j=1}^{n} s_j^2\right)\right] . \qquad (1.1)$$

In addition, we have computed a smaller number of terms for the one-component two-dimensional model. All these calculations of perturbation-expansion terms are described in detail in a separate paper.[11] In the present paper, we detail the analysis of the quantitative implications of these coefficients. In addition, recently using the method of Lipatov,[12] Brézin *et al.*[13] have computed the asymptotic behavior of the perturbation theory terms for large order. They refine our previous estimate[7] of $f_n \propto (n!)$ to

$$f_n \propto n!\, a^n n^b \qquad (1.2)$$

with explicit values of a and b. We have incorporated this information into our analysis and find that it improves significantly the apparent accuracy of our results.

We find, as a result of what we feel to be careful and extensive analysis of the series data, the following results for three dimensions (more detailed results are presented in Table XII)

$$\begin{aligned} n&=0, \quad \nu=0.588\pm 0.001, \quad \gamma=1.161\pm 0.003\,;\\ n&=1, \quad \nu=0.630\pm 0.002, \quad \gamma=1.241\pm 0.004\,;\\ n&=2, \quad \nu=0.669\pm 0.003, \quad \gamma=1.316\pm 0.009\,;\\ n&=3, \quad \nu=0.705\pm 0.005, \quad \gamma=1.39\pm 0.01\,; \end{aligned} \qquad (1.3)$$

where ν is the correlation length index $\xi \propto (T - T_c)^{-\nu}$ and γ is the magnetic susceptibility index $\chi \propto (T - T_c)^{-\gamma}$. The central estimates for $n=1$, the continuous Ising model, are not significantly different from those we had obtained previously,[7] and maintain the small difference with high-temperature series results[14] which do not appear to obey hyperscaling, a property which holds automatically for Callan-Symanzik theory.[6] For $n=2$, the classical XY model with a distribution of spin lengths, the results (1.3) do not differ significantly from the high-temperature series results for the classical XY model. Nor for $n=3$, the classical Heisenberg model with a distribution of spin lengths, is there a significant difference between our results and the corresponding high-temperature series results for the classical Heisenberg model. In the case of the polymer $n=0$, there is a small discrepancy in the value of ν between our results and the corresponding high-temperature series results, but those high-temperature series results do not appear to satisfy hyperscaling, although the failure is not very large compared to the apparent error.

Our results for two dimensions are of a more preliminary nature, owing to the slower convergence and shorter series. Here we find

$$n=1, \quad \nu=0.9\pm0.3, \quad \gamma=1.7\pm0.2\,, \tag{1.4}$$

which are to be compared with the exactly known Ising-model values of $\nu=1$ and $\gamma=\frac{7}{4}$.

In Sec. II of this paper, we introduce our method of analysis and discuss its characteristics. This method is basically the Borel-Leroy, Padé summation method. We list the series data to be used. In the third section we illustrate, and test this method on Wilson's approximate recursion relations. Here, a perturbation series of a similar character is generated for the solution of a nonlinear integral equation. Since this integral equation can be solved to high accuracy, numerically, we can compare the results of our series analysis and error assessment with the correct results. We find that our method of analysis is effective in this case and that our error assessments are realistic.

In Sec. IV we present our analysis of the series data for the n-component $(\phi^2)^2$ model and our results.

II. METHOD OF SERIES ANALYSIS

The derivation of all the series considered in this paper has been described in detail in a separate paper.[11] For the convenience of the reader, we list here the series to be considered. The model for which these series were computed is defined by the partition function,

$$Z=\int\cdots\int\Big(\prod_{\vec{q}} d\vec{S}_{\vec{q}}\Big)\exp\Big(-\frac{1}{2}\int d\vec{k}(m_0^2+k^2)\vec{S}_{\vec{k}}\cdot\vec{S}_{-\vec{k}}-\frac{1}{4!}u_0\int\int\int d\vec{k}_1 d\vec{k}_2 d\vec{k}_3(\vec{S}_{\vec{k}_1}\cdot\vec{S}_{\vec{k}_2})(\vec{S}_{\vec{k}_3}\cdot\vec{S}_{-\vec{k}_1-\vec{k}_2-\vec{k}_3})\Big) \tag{2.1}$$

where the $\vec{S}_{\vec{k}}$ are n-component vectors defined over a d-dimensional space. The series obtained are for the coefficients of the corresponding Callan-Symanzik equation

$$\Big(m\frac{\partial}{\partial m}+\beta(u)\frac{\partial}{\partial u}-(\tfrac{1}{2}N-L)\eta(u)-L\eta_4(u)\Big)\Gamma^{(L,N)}$$
$$=\Delta\Gamma^{(L,N)}\simeq 0\,, \tag{2.2}$$

where the $\Gamma^{(L,N)}$ are the N-point, vertex functions with Ls^2 insertions, or one-particle, irreducible Green's functions. Analysis of the Callan-Symanzik equation[6] leads to the following prescription to compute the critical indices of the model (2.1). First compute the smallest positive value u^*, such that $\beta(u^*)=0$. Then we have

$$\eta=\eta(u^*), \quad \nu^{-1}-2+\eta=\eta_4(u^*)\,, \tag{2.3}$$

where ν is the exponent of divergence for the analogue in this model of the correlation length in the Ising model, $\xi\propto(T-T_c)^{-\nu}$, and η is the analogue of the low-frequency, magnetic-susceptibility index, $\chi\propto k^{\eta-2}$ at $T=T_c$.

We have, for $d=3$, the series

$$\beta(v)=-v+v^2-0.422\,496\,5707v^3+0.351\,069\,5978v^4-0.376\,526\,8283v^5+0.495\,547\,51v^6-0.749\,689v^7+\cdots\,,$$
$$\eta(v)=0.010\,973\,9369v^2+0.000\,914\,2223v^3+0.001\,796\,2229v^4-0.000\,653\,70v^5+0.001\,387\,8v^6-\cdots\,, \tag{2.4}$$
$$\eta_4(v)=-\tfrac{1}{3}v+\tfrac{2}{27}v^2-0.044\,310\,2531v^3+0.039\,519\,5689v^4-0.044\,400\,36v^5+0.060\,3632v^6-\cdots$$

for $n=1$. We have used the change of variable $u=48\pi v/(n+8)$ and $\beta(v)=(n+8)\beta(u)/48\pi$ to define a convenient numerical scale in which the first two coefficients of $\beta(v)$ are -1 and $+1$. It is to be noted that we have improved the accuracy and extended by one term these results over our previous work.[7]

In addition, for $d=3$ and $n=0$ we now have

$$\beta(v)=-v+v^2-0.439\,814\,8149v^3+0.389\,922\,6895v^4-0.447\,316\,0967v^5+0.633\,855\,50v^6-1.034\,928v^7+\cdots\,,$$
$$\eta(v)=0.009\,259\,2593v^2+0.000\,771\,3750v^3+0.001\,589\,8706v^4-0.000\,660\,62v^5+0.001\,4103v^6+\cdots\,, \tag{2.5}$$
$$\eta_4(v)=-\tfrac{1}{4}v+\tfrac{1}{16}v^2-0.035\,767\,2729v^3+0.034\,374\,8465v^4-0.040\,895\,86v^5+0.059\,7048v^6+\cdots\,,$$

and for $n=2$,

$$\begin{aligned}
\beta(v) &= -v + v^2 - 0.402\,962\,9630v^3 + 0.314\,916\,9420v^4 - 0.317\,928\,48v^5 + 0.391\,1025v^6 - 0.552\,449v^7 + \cdots\,, \\
\eta(v) &= 0.011\,851\,8519v^2 + 0.000\,987\,3601v^3 + 0.001\,836\,8107v^4 - 0.000\,586\,33v^5 + 0.001\,2514v^6 - \cdots\,, \\
\eta_4(v) &= -\tfrac{2}{5}v + \tfrac{4}{50}v^2 - 0.049\,513\,4445v^3 + 0.040\,788\,1056v^4 - 0.043\,761\,96v^5 + 0.055\,5573v^6 - \cdots\,,
\end{aligned} \tag{2.6}$$

and for $n=3$,

$$\begin{aligned}
\beta(v) &= -v + v^2 - 0.383\,226\,2015v^3 + 0.282\,946\,6813v^4 - 0.270\,333\,30v^5 + 0.312\,5559v^6 - 0.414\,861v^7 + \cdots\,, \\
\eta(v) &= 0.012\,243\,6486v^2 + 0.001\,020\,0001v^3 + 0.001\,791\,9258v^4 - 0.000\,504\,10v^5 + 0.001\,0883v^6 - \cdots\,, \\
\eta_4(v) &= -\tfrac{5}{11}v + \tfrac{10}{121}v^2 - 0.052\,551\,9563v^3 + 0.039\,964\,0006v^4 - 0.041\,322\,00v^5 + 0.049\,0927v^6 - \cdots\,.
\end{aligned} \tag{2.7}$$

For $d=2, n=1$ we have the series

$$\begin{aligned}
\beta(v) &= -v + v^2 - 0.716\,1736v^3 + 0.930\,7665v^4 - 1.582\,3882v^5 + \cdots\,, \\
\eta(v) &= 0.033\,966v^2 - 0.002\,023v^3 + 0.011\,393v^4 - \cdots\,, \\
\eta_4(v) &= -\tfrac{2}{3}v + 0.250\,047v^2 - 0.233\,588v^3 + 0.323\,089v^4 - \cdots\,,
\end{aligned} \tag{2.8}$$

where we have used the change of variables $u = (8\pi/3)v$ and $\beta(v) = (3/8\pi)\beta(u)$ here.

In addition to these series, we have also the corresponding series for the Wilson-Fisher approximate recursion formula.[15] Again the derivation of these series is discussed in detail in a separate paper.[11] This set of recursion relations is

$$\begin{aligned}
I_\mu(x) &= \int_{-\infty}^{+\infty} \exp[-Ky^2 - \tfrac{1}{2}Q_\mu(x+y) - \tfrac{1}{2}Q_\mu(x-y)]\,dy\,, \\
Q_{\mu+1}(x) &= -b^d \ln[I_\mu(b^{(2-d)/2}x)/I_\mu(0)]\,.
\end{aligned} \tag{2.9}$$

For $b = 2^{1/d}$, Baker[16] has shown that (2.9) is equivalent to a long-ranged, continuous-spin Ising model on a hypercubical lattice in d dimensions with a structure equivalent to that of Dyson's hierarchical model.[17] We have evaluated the series for $b = 1.460\,7378$. This value of b was chosen so that ratio of the first two coefficients would equal that in the exact series. Thus using the same normalization as in (2.4), we have

$$\begin{aligned}
\beta(v) = {} & -v + v^2 - 0.444\,444\,445\,282v^3 + 0.374\,393\,946\,26v^4 - 0.429\,385\,5887v^5 + 0.603\,655\,7599v^6 \\
& - 0.984\,533\,962\,84v^7 + 1.806\,761\,580v^8 - 3.659\,017\,195v^9 + 8.072\,116\,37v^{10} \\
& - 19.220\,722\,64v^{11} + 49.062\,1493v^{12} - 133.545\,1085v^{13} + 385.992\,7975v^{14} + \cdots \\
\eta(v) = {} & 0.0\,, \\
\eta_4(v) = {} & -\tfrac{1}{3}v + 0.074\,074\,074\,2136v^2 - 0.052\,004\,960\,587v^3 + 0.046\,789\,741\,92v^4 \\
& - 0.058\,024\,336\,62v^5 + 0.084\,352\,578\,67v^6 - 0.141\,673\,0744v^7 \\
& + 0.265\,534\,7624v^8 - 0.547\,312\,744v^9 + 1.225\,383\,755v^{10} \\
& - 2.955\,454\,70v^{11} + 7.629\,863\,88v^{12} - 20.979\,651\,98v^{13} + \cdots
\end{aligned} \tag{2.10}$$

In view of the similarity of (2.10) and (2.4), and since highly accurate numerical solutions of the integral equations (2.9) are available, it is of interest to analyze (2.10) by our methods.

In order to effectively analyze all these series, it is useful to understand their structure. As we pointed out previously,[7] it follows from the general theory of graphs with four lines joining at each vertex that we expect of the order of $(2j)!$ graphs in jth order, but that the contribution (except for a certain less numerous subset of graphs) of each graph will be of order $1/(j!)$. Thus we expect the series for β, η, and η_4 to diverge like $j!$ In fact, more can be said. Brézin *et al.*[13] have found that in these series the jth coefficients diverge proportional to

$$(j)!\,(-a)^j\, j^{(3+d+n)/2}\,, \tag{2.11}$$

where $a > 0$ has been explicitly calculated. We will incorporate this behavior into our analysis, and leave to Le Guillou and Zinn-Justin[18] the exploitation of their additional information in a variety of ways. Their analysis depends explicitly on the special hypothesis that $B(x)$ [Eq. (2.12)] is analytic in the whole cut, complex plane.

Since these series are only asymptotic and not convergent, it is necessary to take this fact into account when computing approximations to the function that they represent. We have chosen the Padé-Borel-Leroy method[19] of summation. This method is as follows. First the Borel-Leroy method of summability is based on the formula

$$A(x)=\sum_{n=0}^{\infty} a_n x^n=\int_0^{\infty} t^b e^{-t}\left(\sum_{n=0}^{\infty} \frac{a_n(xt)^n}{\Gamma(n+b+1)}\right) dt\,,$$
$$=\int_0^{\infty} t^b e^{-t} B(xt)\, dt\,. \qquad (2.12)$$

If the coefficients a_n diverge like those in (2.11) then the function $B(xt)$ has a Taylor series expansion with nonzero radius of convergence. The Borel-Leroy method is then to continue analytically $B(y)$ for $0 \leqslant y < \infty$ and to perform the integral in (2.12). If the singularity at the origin of $A(x)$ which gives rise to (2.11) is the only singularity, then the Borel-Leroy method will sum $A(x)$ in the cut-plane $-\infty \leqslant x < 0$. If there are other singularities of $A(x)$, not at $x=0$, then the method is limited to the Borel polygon of summability. This polygon is constructed by drawing a line through each singularity, perpendicular to the ray from it to the origin. The smallest closed polygon of the cut-plane containing the origin is then the Borel polygon of summability. In order to make the analytic continuation required by the Borel-Leroy method, we use the Padé-approximant method[20] on the series for $B(y)$.

The Padé approximant method is defined by the equations

$$L/M = P_L(y)/Q_M(y)\,,$$
$$Q_M(y)B(y) - P_L(y) = O(y^{L+M+1})\,, \qquad (2.13)$$
$$Q_M(0)=1.0$$

where P_L and Q_M are polynomials of degree at most L and M, respectively. We then replace $B(xt)$ in (2.12) by the Padé approximant,

$$A(x) \simeq \int_0^{\infty} t^b e^{-t}\left(\frac{P_L(xt)}{Q_M(xt)}\right) dt\,. \qquad (2.14)$$

In order to perform this integral we decompose the Padé approximant in rational fractions as

$$\frac{P_L(xt)}{Q_M(xt)}=\sum_{j=0}^{L-M} \gamma_j (xt)^j + \sum_{k=1}^{M} \frac{\alpha_k}{xt-\beta_k}\,, \qquad (2.15)$$

where the sum over j is omitted if $L-M<0$. When (2.15) is substituted in (2.14) we have

$$A(x) \simeq \sum_{j=0}^{L-M} \Gamma(j+b+1)\gamma_j x^j$$
$$-\sum_{k=1}^{M} \frac{\alpha_k}{\beta_k} \int_0^{\infty} \frac{t^b e^{-t}\, dt}{1-\beta_k^{-1} x t} \qquad (2.16)$$

which expresses the approximation in terms of the poles and residues of the Padé approximation to $B(y)$ and the hypergeometric function

$${}_2F_0(\alpha,1;x)=\frac{1}{\Gamma(\alpha)}\int_0^{\infty} \frac{t^{\alpha-1}e^{-t}\, dt}{1-tx}\,. \qquad (2.17)$$

It will be noted that in the case where β_k is real and nonnegative approximation (2.16) is strictly speaking undefined. In this case in order to assign a value, since the correct answer is real for our problem, we have chosen the principal value of the integral. This procedure is implemented practically by replacing the integral in (2.16) by a numerical integration along a parabolic contour in the complex $t=r+is$ plane described by

$$s^2 = 0.36r \qquad (2.18)$$

which happens to separate all the poles for our cases in the correct manner. The contour is traversed from small to large r on both (s is positive and negative) branches.

From the theoretical point of view, we would like to define $A(x)$ of (2.12) by the requirement that the formal series be asymptotic in the closed right-half plane. The reason for this requirement is that this requirement has been rigorously proven for three-dimensional ϕ^4 field theory[21] and given the behavior (2.11) of the coefficients, Carleman's theorem[22] proves that there is then at most one such function. As long as β_k in the representation (2.16) is not real and positive, we can demonstrate this asymptotic property for approximation (2.16) by the use of Cauchy's theorem to rotate the contours of integration (separately) in Eq. (2.16). We can equally well choose any path $t=Te^{i\theta}$, $0 \leqslant T \leqslant \infty$ with $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ without changing the value of the integral. In this case, that rotation is sufficient to move all the cuts of the hypergeometric functions in our approximation (2.16) into the left-half plane.

The case when a β_k is real and positive is much different. Consider the function

$$E(z)=\int_0^{\infty} \frac{e^{-t}\, dt}{1-zt} = 1+1!z+2!z^2+3!z^3+\cdots\,. \qquad (2.19)$$

This function is well defined in the cut z plane $0 \leqslant z \leqslant +\infty$, or by contour rotation, with any cut $z=Te^{i\theta}$, $0 \leqslant T \leqslant \infty$, $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. Thus, it can be defined in the wedge $-\frac{1}{2}\pi < \arg z < \frac{5}{2}\pi$, a range of 3π in angle. Neither of these definitions for z real and positive is real, however they are complex conjugate of each other. The principal value integral is their average $\frac{1}{2}[E(x)+E(xe^{2\pi i})]$. This function is defined only in the range $|\arg x| < \frac{1}{2}\pi$. This range falls just short of that needed for a unique definition, i.e., the imaginary axis is also needed. That this problem is real in this example can be easily seen, as the difference between the principal value integral and (2.19) is just

$$\pm \frac{i\pi}{x} e^{-1/x} = 0 + Ox + Ox^2 + \cdots . \qquad (2.20)$$

as x is of arg0 or 2π. Since (2.20) approaches the origin in the manner of infinitely rapid oscillations of diverging amplitude when $x \to 0$ in a purely imaginary manner, and since $E(z)$ is asymptotic for $z \propto e^{i\pi/2}$, it follows that the principal value integral is not. In fact, for this example there are no solutions which are asymptotic in the whole angle $|\arg(x)| \le \frac{1}{2}\pi$ and an infinite number in the angle $|\arg(x)| < \frac{1}{2}\pi$ as can be seen by adding $Ke^{-1/x}$ to the principal value integral. This solution is however the only real function definable from the integral representation (2.19) for positive real z.

In this paper, we take the view that when a pole appears on the positive real axis, which situation contradicts the rigorous results on the Borel summability of field theory in three-dimensions, it is either a reflection of a lack of local convergence, or an approximation to a pair of poles or other singularities symmetrically placed off the axis, or even possibly a reflection of exponential growth in the positive real direction. This last possibility would limit the range of v for which $\beta(v)$, for example, is summable. In any case, we have systematically excluded the approximations with positive real poles from consideration in our analyses.

III. WILSON'S APPROXIMATE RECURSION RELATIONS

In order to compute the approximations (2.16) described in the previous section, it is necessary to perform, at least approximately, the analytic continuation of the Borel-Leroy transform $B(xt)$. In order to do this effectively it is helpful to determine something about the analytic structure of $B(z)$. For the Wilson approximate recursion relations if we select b of (2.12) in accord with (2.11), that is $b = 3\frac{1}{2}$, then at least the dominant singularity should be a simple pole which is well approximated by Padé approximants. We have performed this calculation with $b = 3\frac{1}{2}$, and also $b = 0$. The features we report for $b = 3\frac{1}{2}$, also appear for $b = 0$, but less sharply. Since for $b = 0$ we expect not pole but a branch point, the ability to locate this singularity is diminished and possible other singularities behind it are obscured.

TABLE I. Coefficient of leading power of z at infinity for $B_\beta(z)$, $b = 3.5$; Wilson's approximate recursion relations.

M	$M+1/M$	$M+2/M$	$M+3/M$
1		0.320	−0.042
2	3.68	0.176	−0.009
3	6.26	0.134	0.003
4	−1.95	0.128	−0.01
5	5.38	0.122	−0.004
6	8.45	0.13	

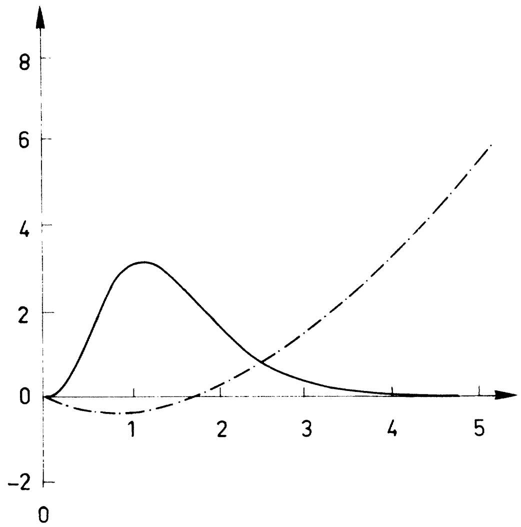

FIG. 1. Broken line is $B_\beta(t)$. For comparison, the solid line is the weight function of Eq. (3.2) which is proportional to $t^{7/2}\exp[-9t/(2v)]$ with $v = 1.5$.

First, in Table I we consider the behavior of $B_\beta(z)$, the Borel-Leroy transform of the function $\beta(z)$ of (2.10) as z goes to infinity. By examination of this table, we see that the coefficient of z^2 from $M+2/M$ is relatively stable and tending to a limit while that for z from $M+1/M$ is large and relatively unstable, and that for z^3 from $M+3/M$ is small and also relatively unstable. The conclusion here is that $B_\beta(z)$ behaves something like z^2 as z goes to infinity so the best Padé approximants will be the ones which mimic this behavior. This asymptotic behavior at infinity agrees with that for the spherical model[6] where $B_\beta(z)$ is exactly a polynomial of degree two. In Fig. 1 we give a plot of $B_\beta(z)$.

Next we seek to explore the singularity structure of $B_\beta(z)$. In Table II we list the location of the closest pole for the same approximants as in Table I. We note that in presenting these results, and subsequent ones that, in order to maintain the normalization of the first two coefficients, we define the Borel-Leroy transform function instead of by (2.12) as

$$B(z) = \sum_{j=0}^{\infty} a_j \frac{\Gamma(b+1)}{\Gamma(b+1+j)} [(b+1)z]^j , \qquad (3.1)$$

so that

TABLE II. Closest pole to the origin for $B_\beta(z)$, $b=3.5$; Wilson's approximate recursion relations.

M	$M+1/M$	$M+2/M$	$M+3/M$
1		−1.9785	−1.6470
2	−1.5611	−1.4140	−1.3733
3	−1.3686	−1.3432	−1.3464
4	−1.3463	−1.3453	−1.3446
5	−1.3445	−1.3421	−1.3453 [a]
6	−1.3452 [a]	−1.3457 [a]	

[a] In these cases, there is a closer pole with very small residue, i.e., a "defect."

$$A(x)=\sum_{j=0}^{\infty} a_j x^j = \frac{(b+1)^{b+1}}{\Gamma(b+1)}\int_0^\infty e^{-(b+1)t}t^b B(tx)\,dt\,. \qquad (3.2)$$

The results reported in Table II, and those from other approximants not tabulated, make it reasonable to conclude that $B_\beta(z)$ has a simple polar singularity at about $z=-1.345\pm 0.003$. This pole has a residue of about +0.35. By the same methods we can observe some further structure. There also appears to be a pole about $z=-2.4\pm 0.1$ with residue about 2.8 and a pole at -5 ± 1 with a residue about +30. Thus the principal elements of structure of the Borel-Leroy transform that emerge are a double pole at infinity and three poles on the negative real axis.

We now proceed to the estimation of the zero of the $\beta(v)$. We have listed in Table III the zeros for the same approximants as in Tables I and II. Based on these entries, plus the Hunter-Baker method of error assessment (Sec. III of Ref. 23) we estimate

$$v^*=1.481\pm 0.001\,. \qquad (3.3)$$

The larger component of the error comes in this case from Hunter-Baker method rather than direct intratabular comparison. The correct answer, obtained by the solution of the integral equation for this problem is

$$v^*=1.4805\,, \qquad (3.4)$$

which is quite compatible with our estimate (3.3).

If instead of using the full length of these series to estimate the location of the zero, we use only the same number of terms as we have available for the exact series (entries above the dashed line in Table III) then we estimate

$$v^*=1.483\pm 0.005\,, \qquad (3.5)$$

which is five times further than (3.3) with an apparent error also five times greater. Since Brézin *et al.*[13] have also computed the value of a for the exact case in (2.11) as well as b, let us next investigate how the inclusion of this information (location of the nearest pole) changes the results. In Table IV we have listed the results when two-point Padé-approximants to $B_\beta(z)$ are used. In addition to the series expansion about $z=0$, we require $B_\beta(-1.34)=\infty$. This additional information allows the calculation of approximants as though we had eight instead of seven coefficients. This method leads to the estimate

$$v^*=1.482\pm 0.005\,, \qquad (3.6)$$

which is an improvement, but not a substantial one, over (3.5). Since both methods are of comparable accuracy for the length of series available, we have in fact performed the computations by both methods for the exact series.

Another quantity of interest is Wegner's correction to scaling index[24]

$$\omega=\beta'(v)\,. \qquad (3.7)$$

We estimate it by procedures similar to those used in Table III to estimate v^*, we find from analysis based on $b=3.5$, that

$$\omega=0.740\pm 0.002 \qquad (3.8)$$

as compared to the correct answer $\omega=0.7425$ obtained from the solution of the integral equation.

We now turn to an analysis of the other series $\eta_4(v)$. Using the Borel-Leroy transform ($b=3.5$), we first investigate the behavior at infinity. We

TABLE III. Zeros of $\beta(v)$ for the Wilson's approximate recursion relation for $b=3.5$.

M	$M+1/M$	$M+2/M$	$M+3/M$
1		1.426 75	1.531 43
2	1.497 75	1.478 64	1.483 79
3	1.482 99	1.481 58	1.481 36
4	1.481 35 [b]	1.481 98 [a]	1.480 95 [b]
5	1.480 87 [b]	1.481 29 [a]	1.480 44 [a,b]
6	1.480 44 [a,b]	1.481 06 [a]	

[a] In these cases there is a close pole and zero, i.e., a defect. It may be that the entry is anomalously close to the one above it.

[b] In these cases there is a pole on the positive axis. All are further from the origin than 5.

TABLE IV. Zeros of $\beta(v)$ for the Wilson's approximate recursion relation for $b=3.5$ and $B_\beta(-1.34)=\infty$.

M	$M+1/M$	$M+2/M$	$M+3/M$
1			1.740 88
2	1.524 75	1.473 51	1.486 76
3	1.484 37	1.481 51	

TABLE V. $\eta_4(1.48)$ for the Wilson's approximate recursion relations for $b=3.5$.

M	$M-1/M$	M/M	$M+1/M$	$M+2/M$	$M+3/M$	$M+4/M$
1				−0.406 45	−0.416 62	−0.412 33
2		−0.406 74	−0.409 48	−0.413 60	−0.415 11	−0.416 47
3	−0.414 33	−0.414 51	−0.417 59 [b]	−0.418 02 [b]	−0.418 29 [b]	−0.419 48 [b]
4	−0.414 32 [a]	−0.418 89 [b]	−0.419 32 [b]	−0.419 32 [b]	−0.416 95	−0.419 31 [a,b]
5	−0.419 29 [b]	−0.419 32 [b]	−0.419 32 [a,b]	−0.419 30 [b]	−0.419 31 [b]	
6	−0.419 29 [a,b]	−0.419 29 [b]	−0.419 32 [b]			
7	−0.419 30 [b]					

[a] In these cases there is a close pole and zero, i.e., a defect. It may be that the entry is anomalously close to the one above it.

[b] In these cases there is a pole on the positive real axis at a distance of about 5 with a residue of magnitude-of-order unity.

find poor convergence for $B_{\eta_4}(z)$ beyond z around 5. There are however some indications that $B_{\eta_4}(z)$ is smaller at infinity than z^4 and larger than z^{-1}. Generally the same poles appear, but with less accuracy, on the negative real axis as was the case for $B_\beta(z)$. We list in Table V the values of $\eta_4(v^*)$, where we have simply used $v^*=1.48$. As will be observed from this table, there is apparent convergence to a value of

$$\eta_4(v^*) = -0.4193 \pm 0.002\,, \tag{3.9}$$

where the largest contribution to the error here is the intratabular variation. It may, however, be noted that, except for the $\frac{7}{4}$, all the higher-order approximants have obvious structural deficiencies, such as poles on the positive real axis, which are excluded by rigorous results[21] for the exact three-dimensional ϕ^4 theory. The solution of the integral equation for the Wilson's approximate recursion relation gives ($\nu=0.6331, \eta=0$)

$$\eta_4(v^*) = -0.4203\,. \tag{3.10}$$

If we restrict our attention to only the number of coefficients that we have available for the exact series in three-dimensions, (above the dotted line in Table V) then we estimate

$$\eta_4(v^*) = -0.414 \pm 0.009\,, \tag{3.11}$$

where the largest contribution to the error comes from the Hunter-Baker projections. If we include the information on the pole location, then from Table VI we estimate

$$\eta_4(v^*) = 0.414 \pm 0.005\,, \tag{3.12}$$

where again the Hunter-Baker error terms are largest. Note that (3.10) lies slightly, but not significantly outside this error estimate.

We conclude that our methods are adequate to analyze the series for the Wilson's approximate recursion relations and to give reasonable estimates of the error of estimation. We proceed to the analysis of the exact series in Sec. IV.

IV. n-COMPONENT $(\phi^2)^2$ THEORY

In this section we analyze the series for the Callan-Symanzik equation coefficients quoted in the second section by the methods which we explored in detail in Sec. III for Wilson's approximate recursion relations. We will discuss the case $d=3$, $n=1$ in detail and summarize the remaining cases.

First we have considered the location of the zero of the function $\beta(v)$. Using the Padé-Borel method of Ref. 7 ($b=0$) we obtain the estimate $v^*=1.42 \pm 0.03$. The more conservative error estimate quoted here, in spite of having an additional series term represents a more thorough analysis of the errors of estimation than was possible at that time. The central value is however unchanged. If we include the information on the nature of the

TABLE VI. $\eta_4(1.48)$ for the Wilson's approximate recursion relations for $b=3.5$ and a pole fixed at -1.34.

M	$M-1/M$	M/M	$M+1/M$	$M+2/M$	$M+3/M$	$M+4/M$
1					−0.423 02	−0.405 25
2			−0.408 83	−0.412 27	−0.414 77	
3	−0.415 41	−0.412 84	−0.412 33			
4	−0.414 38					

TABLE VII. Zero v^* of $\beta(v)$ for the case $n=1$, $d=3$, $b=3.5$ with a pole at $-1.503\,79$.

M \ L	2	3	4	5	6
1			1.546 93	1.361 66	
2		1.466 30	1.423 60	1.416 46	1.415 77
3	... [a]	1.407 13 [b]	1.415 48 [b]	1.415 68 [b]	
4	... [a]	1.416 93	1.415 69		
5		1.415 49 [b]			

[a] No positive real zero.
[b] Positive real poles.

singularity obtained by Brézin *et al.*[13] ($b=3.5$) then the estimate of the location of the zero improves to $v^*=1.416\pm 0.005$. If we include all the information (location of the closest singularity at $-1.503\,79$ in our units), then we obtain the estimate

$$v^*=1.416\pm 0.0015\,. \tag{4.1}$$

This estimate is based on the data tabulated in Table VII. The largest error is estimated by the Hunter-Baker method in this case. It will be observed that the incorporation of the location and nature of the closest singularity has reduced the apparent errors of estimation by a factor of 20 in this case. The approximants with positive real poles have been excluded from our analysis.

In addition from these approximants we may estimate the value of $\omega=\beta'(v^*)$, Wegner's correction to scaling index. We find

$$\omega=0.788\pm 0.003 \tag{4.2}$$

from the data in Table VIII.

Next we analyze the series for η_4. We find

$$\eta_4=1/\nu-2+\eta=-0.382\pm 0.005\,. \tag{4.3}$$

Here it is to be noted that our fuller analysis has slightly widened, despite more information, the uncertainty in η_4. However the central estimate is consistent with our previous one. The error due to the uncertainty in v^* is small compared to the Hunter-Baker error in this case. The data is given in Table IX.

TABLE VIII. ω for the case $n=1$, $d=3$, $b=3.5$ with a pole at $-1.503\,79$.

M \ L	2	3	4	5	6
1			0.627 16	0.885 79	
2		0.728 78	0.775 62	0.786 69	0.788 12
3	... [a]	0.804 37 [b]	0.788 83 [b]	0.788 38 [b]	
4	... [a]	0.785 42	0.788 35		
5		0.788 94 [b]			

[a] No positive real zero.
[b] Positive real pole.

TABLE IX. $\eta_4(v^*)$ for the case $n=1$, $d=3$, $b=3.5$ with a pole at $-1.503\,79$.

M \ L	2	3	4	5
1			−0.389 37	−0.378 92
2		−0.384 45	−0.383 47	−0.382 31
3	−0.385 50 [a]	−0.383 58	−0.383 54 [a]	
4	−0.382 50	−0.382 02		

[a] Pole on the positive real axis.

The direct estimate of η from its series terms has lead to a table which has so many positive poles as to make interpretation impossible. We have chosen instead to analyze the series for $\eta_4-\eta$. Since $\eta\equiv 0$ for the Wilson's approximate recursion relations we have not been able to verify these procedures there. We conclude that

$$\eta_4-\eta=-0.413\pm 0.006 \tag{4.4}$$

or

$$\eta=0.031\pm 0.011 \tag{4.5}$$

on the basis of the results in Table X.

While we can compute from the scaling relations, which hold exactly for this model, the value of γ in terms of η and η_4, an estimate with a slightly smaller error results from the direct estimation of $(1/\gamma-1)=\eta_4/(2-\eta)$. We obtain

$$1/\gamma-1=-0.1939\pm 0.003 \tag{4.6}$$

or

$$\gamma=1.241\pm 0.004 \tag{4.7}$$

again, the fuller analysis leading to a wider error estimate than before.[7] The data is tabulated in Table XI.

We are able to derive from these direct estimates several other indices of interest by the scaling relations

TABLE X. $(\eta_4-\eta)(v^*)$ for the case $n=1$, $d=3$, $b=3.5$ with a pole at $-1.503\,79$.

M \ L	2	3	4	5
1			−0.416 69	−0.409 44
2		−0.406 74	−0.411 58	−0.412 76
3	−0.413 42 [a]	−0.412 56 [a]	−0.414 31 [a]	
4	−0.412 50 [a]	−0.413 05 [a]		

[a] Pole on the positive real axis.

TABLE XI. $(1/\gamma - 1)$ for the case $n=1$, $d=3$, $b=3.5$ with a pole at −1.503 79.

M \ L	2	3	4	5
1			−0.197 09	−0.192 06
2		−0.193 71	−0.194 00	−0.193 92
3	−0.194 80 [a]	−0.194 02 [a]	−0.193 94	
4	−0.193 82	−0.193 89		

[a] Pole on the positive real axis.

$$
\begin{aligned}
&\alpha = 2 - d\nu = 0.110 \pm 0.008\,,\\
&\beta = \tfrac{1}{2}(d\nu - \gamma) = 0.324 \pm 0.006\,,\\
&\delta = (d+2-\eta)/(d-2+\eta) = 4.82 \pm 0.06\,,\\
&\Delta = \tfrac{1}{2}(d\nu + \gamma) = 1.566 \pm 0.006\,,\\
&\nu = (2+\eta_4-\eta)^{-1} = 0.630 \pm 0.0025\,.
\end{aligned}
\tag{4.8}
$$

In exactly the same way we have analyzed the series for $n=0$, 2, 3, as well. Our results are summarized in Table XII. It is of interest to compare these results with those which have been previously obtained by high-temperature series methods for the corresponding fixed spin-length models (Ising, etc....). The most complete comparison is possible with the $s=\frac{1}{2}$ Ising model.[25] There is a difference with this model in that there hyperscaling fails, i.e.,

$$2\Delta - d\nu - \gamma = -0.028 \pm 0.003 \tag{4.9}$$

instead of zero as in (4.8). If hyperscaling is forced, then one has[14,25] as the best results ($v^*=0$ without this assumption)

$$
\begin{aligned}
&v^* = 1.46 \pm 0.02,\quad \nu = 0.638 \pm 0.001\,,\\
&\gamma = 1.250 \pm 0.003,\quad \Delta = 1.563 \pm 0.003\,,
\end{aligned}
\tag{4.10}
$$

which are, while not quite in agreement with Table XII, not so dramatically different.

For the polymer case, series analysis gives[26-29]

$$\gamma = 1.165 \pm 0.003 \quad \nu = 0.600 \pm 0.005\,. \tag{4.11}$$

There is a small discrepancy here in ν, but not a very large one. It should be remarked[26] that using the data for α the series analysis[30,31] leads to

$$2 - \alpha - d\nu = -0.05 \pm 0.03\,, \tag{4.12}$$

which also suggests that hyperscaling is violated in this case as well as for $n=1$.

For the cases $n=2$ and 3, the series results are[32]

$$\gamma = 1.318 \pm 0.01,\quad \nu = 0.670 \pm 0.006\,, \tag{4.13}$$

for $n=2$, and[33,34]

$$\gamma = 1.375^{+0.02}_{-0.01},\quad \nu = 0.70 \tag{4.14}$$

for $n=3$. These results are consistent within the respective quoted errors with ours as listed in Table XII. We conclude that all the apparent differences between the n component, $(\phi^2)^2$ model and the fixed spin-length model are contained in the variation of the critical parameters necessary to accomodate hyperscaling in this model, and its numerically small, apparent violation in the fixed spin-length model.

Finally, we have analyzed the series for the two-dimensional, one-component model by the same methods as we have used above. Here, because the known series is both shorter and the location of the apparent zero in units of the pole distance is considerably further from the origin, the results are not so good. If we use the Padé-Borel method of Ref. 7 ($b=0$), we obtain the estimate $v^*=1.8 \pm 0.2$. If we next include the information on the nature of the singularity obtained by Brézin *et al.*[13] ($b=3$), then the estimate becomes 1.9 ± 0.3. Finally using all the information (the location of the closest singularity is −1.047 52 in our units), then we obtain the estimate

$$v^* = 1.8 \pm 0.3\,, \tag{4.15}$$

which we adopt. This result is to be compared

TABLE XII. Critical properties of the ϕ^4 model in $d=3$.

	$n=0$	$n=1$	$n=2$	$n=3$
v^*	1.421 ± 0.004	1.416 ± 0.0015	1.406 ± 0.005	1.392 ± 0.009
ω	0.794 ± 0.006	0.788 ± 0.003	0.78 ± 0.01	0.78 ± 0.02
η_4	−0.274 ± 0.01	−0.382 ± 0.005	−0.474 ± 0.008	−0.550 ± 0.012
ν	0.588 ± 0.001	0.630 ± 0.002	0.669 ± 0.003	0.705 ± 0.005
γ	1.161 ± 0.003	1.241 ± 0.004	1.316 ± 0.009	1.39 ± 0.01
α	0.236 ± 0.004	0.110 ± 0.008	−0.007 ± 0.009	−0.115 ± 0.015
β	0.302 ± 0.004	0.324 ± 0.006	0.346 ± 0.009	0.362 ± 0.012
Δ	1.462 ± 0.004	1.566 ± 0.006	1.662 ± 0.009	1.753 ± 0.012
η	0.026 ± 0.014	0.031 ± 0.011	0.032 ± 0.015	0.031 ± 0.022
δ	4.85 ± 0.08	4.82 ± 0.06	4.81 ± 0.08	4.82 ± 0.12

with the high-temperature series result[25] of $v^*=1.751\pm 0.005$. We find, as in (4.2) the value of

$$\omega = 0.7 \pm 0.4\,, \tag{4.16}$$

which is again rather inaccurate. Analyzing the series for η_4, $\eta_4-\eta$, and $\eta_4/(2-\eta)$ we obtain

$$\begin{aligned} &\eta_4 \simeq -0.83\pm 0.2\,,\\ &\eta_4-\eta \simeq -0.91\pm 0.25\,,\\ &\eta_4/(2-\eta)=\gamma^{-1}-1=-0.42\pm 0.07\,,\\ &\gamma=1.72\pm 0.2\,. \end{aligned} \tag{4.17}$$

From these we can derive, as in (4.8), by the scaling relations

$$\begin{aligned} &\alpha=0.16\pm 0.6\,,\quad \beta=0.06\pm 0.4\,,\\ &\delta=5 \text{ to } \infty\,,\quad \Delta=1.78\pm 0.4\,,\\ &\nu=0.92\pm 0.3\,,\quad \eta=0.08\pm 0.2\,. \end{aligned} \tag{4.18}$$

By comparison, if we do not use the location of the singularity or its nature ($b=0$ used) we obtain the estimates

$$\nu=0.9\pm 0.2\,,\quad \gamma=1.7\pm 0.1\,, \tag{4.19}$$

of similar quality to (4.17) and (4.18). We conclude that the information currently available is insufficient to allow precise prediction by our methods in the two-dimensional case, nor can we make a meaningful comparison with the Ising-model results derived by high-temperature series methods.

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